NOTES FOR GEOMETRIC REPRESENTATION THEORY 1

LIN CHEN

Contents

Date: March 5, 2026.

The goal of this course is to study certain representations of complex semisimple Lie algebras, such as $\mathfrak{sl}_2(\mathbb{C})$, via geometric methods.

1. Introduction

1.1. **Recollection: semisimple Lie algebras and root systems.** Let $\mathfrak{g}$ be a complex semisimple Lie algebra. We exclude the case $\mathfrak{g}=0$ to avoid some awkward words.

We fix a Borel subalgebra $\mathfrak{b}\subseteq\mathfrak{g}$ and a Cartan subalgebra $\mathfrak{h}$ contained in $\mathfrak{b}$. Let $\mathfrak{b}^-$ be the opposite Borel subalgebra such that $\mathfrak{b}^-\cap\mathfrak{b}=\mathfrak{h}$. Write $\mathfrak{n}:=[\mathfrak{b},\mathfrak{b}]$ and $\mathfrak{n}^-:=[\mathfrak{b}^-,\mathfrak{b}^-]$.

Recall the following standard terminologies and facts (see [H1]).

- Elements in $\mathfrak{h}^*:=\mathsf{Hom}_{\mathbb{C}}(\mathfrak{h},\mathbb{C})$ are called *weights*.
- Nonzero weights that appear in $\mathfrak{g}$ are called *roots*. We have a *root decomposition*
$$\mathfrak{g}\simeq\mathfrak{h}\bigoplus\Big(\bigoplus_{\alpha\in\Phi}\mathfrak{g}_\alpha\Big),$$
where Φ is the set of roots. Each root subspace $\mathfrak{g}_\alpha$ is 1-dimensional.
- Suppose that α,β and $\alpha+\beta$ are roots, then $[\mathfrak{g}_\alpha,\mathfrak{g}_\beta]=\mathfrak{g}_{\alpha+\beta}$.
- Roots that appear in $\mathfrak{b}$ are called *positive roots*. We have $\Phi=\Phi^+\sqcup-\Phi^+$.
- A positive root is *simple* if it is not the sum of other positive roots. Let $\Delta\subseteq\Phi$ be the subset of positive roots.
- Let $\mathfrak{h}^*_{\mathbb{R}}:=\mathbb{R}\Phi\subseteq\mathfrak{h}^*$ be the $\mathbb{R}$-span of Φ. This gives a real structure on $\mathfrak{h}^*$. In other words, we have $\mathfrak{h}^*_{\mathbb{R}}\otimes_{\mathbb{R}}\mathbb{C}\xrightarrow{\simeq}\mathfrak{h}^*$.
- The *Killing form* of $\mathfrak{g}$ restricts to a non-degenerate symmetric bilinear form on $\mathfrak{h}$. Let $(-,-):\mathfrak{h}^*\times\mathfrak{h}^*\to\mathbb{C}$ be the dual form. Its restriction on $\mathfrak{h}^*_{\mathbb{R}}$ is an inner form.
- The pair $(\mathfrak{h}^*_{\mathbb{R}},\Phi)$ is a *root system*. Let W be the *Weyl group* of this root system, which is the reflection group acting on $\mathfrak{h}^*_{\mathbb{R}}$ with generators
$$s_\alpha(\lambda):=\lambda-2\frac{(\alpha,\lambda)}{(\alpha,\alpha)}\alpha,\ \ \alpha\in\Phi$$
- Elements $s_\alpha\in W$ ($\alpha\in\Delta$) are called *simple reflections*. Recall that W is generated by simple reflections. For $w\in W$, the *length* $l(w)$ of w is the minimal length of writing w as a product of simple reflections. There is a unique longest element in W, which we denote by w_0.
- For each root $\alpha\in\Phi$, we have the corresponding *coroot* $\check{\alpha}\in\mathfrak{h}$ characterized by the formula
$$\check{\alpha}(\lambda)=2\frac{(\alpha,\lambda)}{(\alpha,\alpha)},\ \ \forall\lambda\in\mathfrak{h}^*.$$
Moreover, $[\mathfrak{g}_\alpha,\mathfrak{g}_{-\alpha}]=\mathbb{C}\check{\alpha}$.
- A weight $\lambda\in\mathfrak{h}^*$ is *integral* if $\check{\alpha}(\lambda)\in\mathbb{Z}$ for any root $\alpha\in\Phi$. Integral weights form a $\mathbb{Z}$-lattice $\Lambda\subseteq\mathfrak{h}^*$. In other words, we have $\Lambda\otimes_{\mathbb{Z}}\mathbb{C}\xrightarrow{\simeq}\mathfrak{h}^*$.
- An *integral* weight $\lambda\in\mathfrak{h}^*$ is *dominant* if $\check{\alpha}(\lambda)\geq0$ for any positive root $\alpha\in\Phi^+$. Dominant integral weights form a semigroup $\Lambda^+\subseteq\Lambda$.
- For weights $\lambda,\lambda'\in\Lambda$, we say $\lambda\leq\lambda'$ if $\lambda'-\lambda$ is *positive*. In other words, $\lambda'-\lambda$ is a finite sum of positive roots.

- Let
$$\rho := (\sum_{\alpha\in\Phi^+} \alpha)/2$$
be the half sum of all positive roots. Recall that $\check{\alpha}(\rho) = 1$ for any simple positive root $\alpha \in \Delta$.

1.2. **From finite dimentional representations to Verma modules.** Recall the following well-known result.

Theorem 1.2.1. *The category $\mathfrak{g}\text{−}\mathsf{mod}_{\mathsf{fd}}$ of finite dimensional $\mathfrak{g}$-modules is semisimple. Moreover, there is a canonical bijection*

$$\Lambda^+ \xrightarrow{\simeq} \mathsf{Irr}(\mathfrak{g}\text{−}\mathsf{mod}_{\mathsf{fd}}),\ \lambda \mapsto L_\lambda \tag{1.1}$$

such that L_λ has highest weight λ.

The $\mathfrak{g}$-module L_λ is called the *Weyl module with highest weight* λ.

As an $\mathfrak{h}$-modules, we have

$$\bigoplus_{\mu\le\lambda,\mu\in\Lambda} \mathbb{C}_\mu \otimes \mathsf{Hom}_{\mathfrak{h}}(\mathbb{C}_\mu, L_\lambda) \xrightarrow{\simeq} L_\lambda,$$

where each direct summand in the LHS is a *weight subspace* of L_λ. The following question is natural.

Question 1.2.2. How to calculate the dimension of the μ-weight subspace of L_λ, i.e., $\dim(\mathsf{Hom}_{\mathfrak{h}}(\mathbb{C}_\mu, L_\lambda))$?

Example 1.2.3. For $\mathfrak{g} = \mathfrak{sl}_2$, let $\mathfrak{b}$ and $\mathfrak{h}$ be its standard Borel and Cartan subalgebra. We have short exact sequences

$$0 \to \mathfrak{h} \to \mathbb{C}^2 \xrightarrow{\Sigma} \mathbb{C} \to 0$$

and

$$0 \to \mathbb{C} \xrightarrow{\Delta} \mathbb{C}^2 \to \mathfrak{h}^* \to 0.$$

We identify $\mathfrak{h}^*$ with $\mathbb{C}$ such that $(\lambda_1, \lambda_2) \in \mathbb{C}^2$ is sent to $l := \lambda_2 - \lambda_1$ by the map $\mathbb{C}^2 \to \mathfrak{h}^* \simeq \mathbb{C}$. Note that:

- A weight $\lambda \in \mathfrak{h}^*$ is integral iff the corresponding number l is integral;
- A weight $\lambda \in \mathfrak{h}^*$ is dominant iff the corresponding number l is non-negative;
- For weights $\lambda, \lambda' \in \mathfrak{h}^*$ and the corresponding numbers l, l', we have $\lambda \le \lambda'$ iff $l' - l \in 2\mathbb{Z}^+$.

The vector space L_λ can be identified with the space of homogeneous polynomials $f \in \mathbb{C}[x_1, x_2]$ with degree l, and the $\mathfrak{sl}_2$-action on this vector space is given by changing coordinates. It follows that weights of L_λ correspond to numbers $l - 2d$, $d = 0, \cdots, l$, and each weight subspace is 1-dimensional. Namely, $x_1^{l-d} x_2^d$ is a generator to the weight subspace corresponding to $l - 2d$.

To describe the solution for general $\mathfrak{g}$, we introduce the *character* of $V \in \mathfrak{g}\text{−}\mathsf{mod}_{\mathsf{fd}}$ by the formula

$$\mathsf{ch}(V) := \sum_{\mu\in\mathfrak{h}^*} \dim(\mathsf{Hom}_{\mathfrak{h}}(\mathsf{C}_\mu, V)) T^\mu,$$

which is an element in the polynomial algebra. Note that we have

$$\mathsf{ch}(V \bigoplus V') = \mathsf{ch}(V) + \mathsf{ch}(V'),\ \ \mathsf{ch}(V \otimes V') = \mathsf{ch}(V) \cdot \mathsf{ch}(V').$$

Theorem 1.2.4 (Weyl character formula)**.** *We have*

$$\mathsf{ch}(L_\lambda)\cdot\sum_{w\in W}(-1)^{l(w)}T^{w(\rho)}=\sum_{w\in W}(-1)^{l(w)}T^{w(\lambda+\rho)}.$$

In other words, we can calculate $\mathsf{ch}(L_\lambda)$ by expanding

$$\sum_{w\in W}(-1)^{l(w)}\frac{T^{w(\lambda+\rho)}}{\sum_{w'\in W}(-1)^{l(w')}T^{w'(\rho)}}, \tag{1.2}$$

as a *formal infinite* sum of T^μ ($\mu\in\Lambda$), which turns out to be a finite sum.

Remark 1.2.5. Note that the denominator does not depend on λ.

The following example shows that each term in the alternating sum (1.2) is a genuine infinite sum.

Example 1.2.6. For $\mathfrak{g}=\mathfrak{sl}_2$, we have

$$\mathsf{ch}(L_\lambda)=\frac{T^{l+1}-T^{-l-1}}{T-T^{-1}}=T^l+T^{l-2}+\cdots T^{-l}.$$

However, both

$$\frac{T^{l+1}}{T-T^{-1}}=T^l+T^{l-2}+\cdots$$

and

$$\frac{T^{-l-1}}{T-T^{-1}}=T^{-l-2}+T^{-l-4}+\cdots$$

are infinite sums.

This raises the following natural question.

Question 1.2.7. Is there a representation theoretic intepretation for the term

$$\frac{T^{w(\lambda+\rho)}}{\sum_{w\in W}(-1)^{l(w)}T^{w(\rho)}}?$$

For example, for $\mathfrak{g}=\mathfrak{sl}_2$, we can ask the following question.

Question 1.2.8. Can we find (infinite dimensional) $\mathfrak{sl}_2$-modules $N\subseteq M$ with

$$\mathsf{ch}(M)=\frac{T^{l+1}}{T-T^{-1}},\ \ \mathsf{ch}(N)=\frac{T^{-l-1}}{T-T^{-1}}$$

such that $L_\lambda\simeq M/N$?

It turns out there is a positive answer to Question 1.2.7.

Theorem 1.2.9 (Bernstein–Gelfand–Gelfand resolution)**.** *There is a canonical resolution*

$$0\to M_{w_0\cdot\lambda}\cdots\to\bigoplus_{\ell(w)=2}M_{w\cdot\lambda}\to\bigoplus_{\ell(w)=1}M_{w\cdot\lambda}\to M_\lambda\to L_\lambda\to 0.$$

In above, $M_{w\cdot\lambda}$ is the *Verma module* with highest weight

$$w\cdot\lambda:=w(\lambda+\rho)-\rho,$$

which is an *infinite-dimensional* $\mathfrak{g}$-module.

The upshot is: we need to study infinite-dimensional representations to understand the finite-dimensional ones better.

Warning 1.2.10 (or features)**.** The category $\mathfrak{g}\text{–}\mathsf{mod}$ of (all) $\mathfrak{g}$-modules is not semisimple. A $\mathfrak{g}$-module may fail to be integrable.

Example 1.2.11. As we will soon see, for $\mathfrak{g} = \mathfrak{sl}_2$, the short exact sequence

$$0 \to M_{w_0 \cdot \lambda} \to M_\lambda \to L_\lambda \to 0$$

does not split. Also, M_λ is not an (algebraic) representation for SL_2.

1.3. **Recollection: enveloping algebra.** Recall that for any Lie algebra $\mathfrak{g}$, we have a canonical equivalence

$$\mathfrak{g}\text{–}\mathsf{mod} \simeq U(\mathfrak{g})\text{–}\mathsf{mod}, \tag{1.3}$$

where $U(\mathfrak{g})$ is the *universal enveloping algebra* of $\mathfrak{g}$, which is an associative (but not commutative) $\mathbb{C}$-algebra. Namely, let

$$T(\mathfrak{g}) \coloneqq \bigoplus_{i=0}^{\infty} \mathfrak{g}^{\otimes i}$$

be the tensor algebra for the underlying vector space $\mathfrak{g}$. Consider the two-sided ideal $I \subseteq T(\mathfrak{g})$ generated by elements

$$X \cdot Y - Y \cdot X - [X, Y] \in T(\mathfrak{g}).$$

One can check that

$$U(\mathfrak{g}) \coloneqq T(\mathfrak{g})/I$$

fits into the desired equivalence (1.3).

For future reference, let us recall that the (non-negative) grading on $T(\mathfrak{g})$ induces the *PBW filtration* on $U(\mathfrak{g})$:

$$\mathsf{F}^{\leq n} U(\mathfrak{g}) \coloneqq \mathrm{Im}(\bigoplus_{i=0}^{n} \mathfrak{g}^{\otimes i} \to U(\mathfrak{g})).$$

One can check that this makes $U(\mathfrak{g})$ an *almost commutative filtered associative algebra*, i.e.,

$$\mathsf{F}^{\leq m} U(\mathfrak{g}) \cdot \mathsf{F}^{\leq n} U(\mathfrak{g}) \subseteq \mathsf{F}^{\leq m+n} U(\mathfrak{g})$$

and

$$[\mathsf{F}^{\leq m} U(\mathfrak{g}), \mathsf{F}^{\leq n} U(\mathfrak{g})] \subseteq \mathsf{F}^{\leq m+n-1} U(\mathfrak{g}).$$

In particular,

$$\mathsf{gr}^\bullet U(\mathfrak{g}) \coloneqq \bigoplus_{n=0}^{\infty} \mathsf{gr}^n U(\mathfrak{g}) \coloneqq \bigoplus_{n=0}^{\infty} \mathsf{F}^{\leq n} U(\mathfrak{g}) / \mathsf{F}^{\leq n-1} U(\mathfrak{g})$$

is a graded commutative algebra.

Theorem 1.3.1 (Poincare–Birkhoff–Witt)**.** *The map*

$$\mathfrak{g} \to \mathsf{F}^{\leq 1} U(\mathfrak{g}) \to \mathsf{gr}^1 U(\mathfrak{g}) \to \mathsf{gr}^\bullet U(\mathfrak{g})$$

induces an isomorphism

$$\mathsf{Sym}^\bullet(\mathfrak{g}) \xrightarrow{\simeq} \mathsf{gr}^\bullet U(\mathfrak{g}).$$

The construction $\mathfrak{g} \mapsto U(\mathfrak{g})$ is functorial. In other words, for any Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}'$, we have a ring homomorphism $U(\mathfrak{g}) \to U(\mathfrak{g}')$ such that the following diagram commute

$$\begin{array}{ccc} \mathfrak{g}'\text{–mod} & \xrightarrow{\text{res}} & \mathfrak{g}\text{–mod} \\ \downarrow \simeq & & \downarrow \simeq \\ U(\mathfrak{g}')\text{–mod} & \xrightarrow{\text{res}} & U(\mathfrak{g})\text{–mod}. \end{array}$$

It follows that the top horizontal functor admits a left adjoint

$$\mathrm{ind}_{\mathfrak{g}}^{\mathfrak{g}'} : \mathfrak{g}\text{–mod} \to \mathfrak{g}'\text{–mod}, \quad M \mapsto U(\mathfrak{g}') \underset{U(\mathfrak{g})}{\otimes} M$$

Example 1.3.2. When $\mathfrak{g}' = 0$, we obtain a functor

$$\mathfrak{g}\text{–mod} \to \mathsf{Vect}, \quad M \mapsto \mathbb{C} \underset{U(\mathfrak{g})}{\otimes} M =: M_{\mathfrak{g}}.$$

The vector space $M_{\mathfrak{g}}$ is called the *coinvariance* of M.

1.4. **Verma modules.**

Construction 1.4.1. Consider the following diagram of Lie algebras

$$\begin{array}{ccccc} & & \mathfrak{b} & & \\ & \swarrow & & \searrow & \\ \mathfrak{g} & & & & \mathfrak{b}/[\mathfrak{b},\mathfrak{b}] \xleftarrow{\simeq} \mathfrak{h}. \end{array}$$

For any $\lambda \in \mathfrak{h}^*$, by restriction, we view $\mathbb{C}_\lambda$ as a 1-dimensional $\mathfrak{b}$-module. By induction, we obtain a $\mathfrak{g}$-module

$$M_\lambda := \mathrm{ind}_{\mathfrak{b}}^{\mathfrak{g}}(\mathbb{C}_\lambda) = U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda.$$

Exercise 1.4.2. (Problem 1, Homework 1) Show that

$$U(\mathfrak{g}) \simeq U(\mathfrak{n}^-) \otimes U(\mathfrak{b})$$

as $(U(\mathfrak{n}^-), U(\mathfrak{b}))$-bimodules. Deduce that

$$M_\lambda \simeq U(\mathfrak{n}^-)$$

as $\mathfrak{n}^-$-module.

Exercise 1.4.3. (Problem 2, Homework 1) Show that M_λ is a semisimple $\mathfrak{h}$-module and each weight subspace is finite-dimensional.

In particular, we can talk about *weights* and *weight subspaces* of M_λ.

Exercise 1.4.4. (Problem 3, Homework 1) Show that λ is the unique highest weight of M_λ, and the image of the composition

$$\mathbb{C} \xrightarrow{\subseteq} U(\mathfrak{n}^-) \xrightarrow{\simeq} M_\lambda$$

is the λ-weight subspace.

Definition 1.4.5. We call M_λ the *Verma module* with highest weight λ.

The goal of this course is to study Verma modules and the subcategory $\mathcal{O} \subseteq \mathfrak{g}\text{–mod}$ generated by them.

Exercise 1.4.6. (Problem 4, Homework 1) Deduce Theorem 1.2.4 from Theorem 1.2.9. You can achieve this *without* using the Weyl denominator formula. Hint: find $\mathsf{ch}(U(\mathfrak{n}^-))$ by studying the case $\lambda = 0$.

1.5. **Geometric incarnations.** Let G be an algebraic group with Lie algebra $\mathfrak{g}$. Let $H \subseteq B \subseteq G$ be the Cartan and Borel subgroups corresponding to $\mathfrak{h} \subseteq \mathfrak{b}$. The quotient G/B is a complex variety, called the *flag variety*.

In future lectures, for each character λ of H (which in particular is an integral weight of $\mathfrak{h}$), we will construct a canonical G-equivariant line bundle $\mathcal{O}(\lambda)$ on G/B. Roughly speaking, we consider the trivial line bundle $\mathcal{O}_G$ on G, equipped with the trivial (G,B)-equivariant structure, but twist the right B-equivariant structure by the character λ. By descent theory, this indeed gives a G-equivariant line bundle on G/B.

Example 1.5.1. For $G = \mathsf{SL}_2$, the standard action of G on $\mathbb{A}^2 \setminus 0$ induces a transitive action of G on $\mathbb{P}^1$. The stablizer of $\infty \in \mathbb{P}^1$ for this action is the standard Borel subgroup $B \subseteq G$. It follows that $G/B \simeq \mathbb{P}^1$.

Via this isomorphism, for a character λ with

$$\lambda(\mathsf{diag}(t, t^{-1})) = t^n,$$

the line bundle $\mathcal{O}(\lambda)$ on G/B corresponds to the line bundle $\mathcal{O}(n)$ on $\mathbb{P}^1$.

Theorem 1.5.2 (Borel–Weil–Bott)**.** *For dominant λ, we have a canonical isomorphism of G-modules*

$$\Gamma(G/B, \mathcal{O}(\lambda)) \simeq L_\lambda. \tag{1.4}$$

This motivates the following question.

Question 1.5.3. Is it possible to realize the Verma module M_λ as the global section of some objects over G/B?

We should not restrict ourselves to vector bundles on G/B because there are very few of them. Experiences in algebraic geometry suggest we should try (quasi-coherent) sheaves. However, we should first answer the following preliminary question.

Question 1.5.4. For a quasi-coherent $\mathcal{O}$-module $\mathcal{F}$ over G/B, is there some additional structure on $\mathcal{F}$ that makes $\Gamma(G/B, \mathcal{F})$ a $\mathfrak{g}$-module?

Note that in (1.4), the G-action on the LHS comes from the G-equivariant structure on $\mathcal{O}(\lambda)$. However, the Verma module M_λ is *never* a G-module.

Exercise 1.5.5. (Problem 5, Homework 1) Show that M_λ is not an algebraic representation of G.

Nevertheless, there is a beautiful answer Qusetion 1.5.4, motivated by the following observation.

Theorem 1.5.6. *Let $T_{G/B}$ be the tangent bundle on G/B. We have a canonical isomorphism of Lie algebras:*

$$\Gamma(G/B, T_{G/B}) \simeq \mathfrak{g}.$$

In other words, the Lie algebra of global vector fields on G/B is isomorphic to $\mathfrak{g}$.

We will prove Theorem 1.5.6 in future lectures. For now, please convince yourself using the SL_2-case.

Exercise 1.5.7. (Problem 6, Homework 1) Find an isomorphism of Lie algebras

$$\Gamma(\mathbb{P}^1, T_{\mathbb{P}^1}) \simeq \mathfrak{sl}_2.$$

Theorem 1.5.6 suggests that we should consider sheaves $\mathcal{F}$ over G/B equipped with an action of the tangent sheaf $T_{G/B}$. One can call them *sheaves with connections*, but the standard terminology is *D-modules*. Here the letter "D" stands for "differential".

In future lectures, for each Verma module M_λ, we will attach a (twisted) D-module $\mathcal{F}_\lambda$ on G/B such that

$$\Gamma(G/B, \mathcal{F}_\lambda) \simeq M_\lambda$$

as a $\mathfrak{g}$-modules. Moreover, the BGG resolution (Theorem 1.2.9) will be proved by finding a resolution of $\mathcal{O}(\lambda)$ by the (twisted) D-modules $\mathcal{F}_{w\cdot\lambda}$, which itself is a sheaf theoretic incarnation of the Bruhat decomposition

$$G/B = \bigcup_{w\in W} BwB/B.$$

As a consequence, we obtain a *geometric* intepretation of the Weyl character formula.

The ultimate goal of this course is to relate $\mathfrak{g}$-modules with D-modules on G/B. This is known as the *localization theory* for semisimple Lie algebras.

2. Category $\mathcal{O}$

2.1. Definition of category $\mathcal{O}$.

Definition 2.1.1. We define BGG's category $\mathcal{O}$ to be the full subcategory of $\mathfrak{g}-\mathsf{mod}$ that contains $\mathfrak{g}$-modules M satisfying the following conditions:

(i) As a $\mathfrak{g}$-module, M is finitely generated.
(ii) As an $\mathfrak{h}$-module, M is semisimple. In other words, M is a *weight module*.
(iii) As an $\mathfrak{n}$-module, M is a union of finite-dimensional submodules.

Lemma 2.1.2. *The Verma module M_λ is contained in $\mathcal{O}$.*

Proof. The Verma module M_λ satisfies condition (i) and (ii) respectively by Exercise 1.4.2 and Exercise 1.4.3.

It remains to prove that M_λ satisfies condition (iii). Let v_λ be a nonzero highest weight vector of M_λ. By Exercise 1.4.4,

$$M_\lambda = \bigcup_i \mathsf{F}^i U(\mathfrak{g}) \cdot v_\lambda.$$

Each $\mathsf{F}^i U(\mathfrak{g}) \cdot v_\lambda$ is finite dimensional. Hence we only need to show these subspaces are $\mathfrak{n}$-stable. For $u \in \mathsf{F}^i U(\mathfrak{g})$ and $x \in \mathfrak{n}$ we have

$$x \cdot (u \cdot v_\lambda) = u \cdot (x \cdot v_\lambda) + [x, u] \cdot v_\lambda.$$

Note that $\mathfrak{h}$ acts on $x \cdot v_\lambda$ with weight strictly greater than λ. Hence $x \cdot v_\lambda = 0$. This implies the claim because

$$[x, u] \in [\mathfrak{g}, \mathsf{F}^i U(\mathfrak{g})] \subset \mathsf{F}^i U(\mathfrak{g}) \tag{2.1}$$

□

In fact, we have the following stronger result.

Lemma 2.1.3. *The $\mathfrak{n}$-action on M_λ is locally nilpotent.*

Proof. We only need to show $\mathfrak{n}$ acts nilpotently on each weight vector v of M_λ. This follows from the fact that for $x \in \mathfrak{n}$, the element $x \cdot v$ is a weight vector with weight strictly greater than the weight of v.

□

The following result is obvious.

Lemma 2.1.4. *The subcategory $\mathcal{O} \subseteq \mathfrak{g}-\mathsf{mod}$ is stable under taking finite direct sums and subquotients.*

Corollary 2.1.5. *The category $\mathcal{O}$ is an abelian category.*

Warning 2.1.6. The subcategory $\mathcal{O} \subseteq \mathfrak{g}-\mathsf{mod}$ is *not* stable under taking extensions.

Exercise 2.1.7. (Problem 7, Homework 1) For $\mathfrak{g} = \mathfrak{sl}_2$, find an extension of two Verma modules inside $\mathfrak{g}-\mathsf{mod}$ that is not a weight module. Hint: $\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}$.

Proposition 2.1.8. *Each object in $\mathcal{O}$ is a quotient of a finite successive extension of some Verma modules.*

Proof. Let $M \in \mathcal{O}$ be an object. By condition (i), M is generated as a $\mathfrak{g}$-module by a finite-dimensional subspace $M_0 \subseteq M$. We may enlarge M_0 such that it is a finite dimensional $\mathfrak{h}$-module. In particular, there are finitely many weights in M_0.

Note that each weight in $U(\mathfrak{h})\cdot M_0 = U(\mathfrak{n})\cdot M_0$ is greater or equal to a weight in M_0. Since M is a highest weight module, there are finitely many weights satisfying the above property. Since each weight subspace of M is finite-dimensional, we see that $U(\mathfrak{h})\cdot M_0$ is finite-dimensional. Therefore we can replace M_0 by $U(\mathfrak{h})\cdot M_0$ and assume M_0 is a $\mathfrak{b}$-module.

Now consider the natural $\mathfrak{g}$-linear map

$$\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(M_0) \to M \tag{2.2}$$

induced by the $\mathfrak{b}$-linear map $M_0 \to M$. Since M is generated by M_0, the map (2.2) is surjective. Hence we only need to show $\mathsf{ind}_{\mathfrak{b}}^{\mathfrak{g}}(M_0)$ is a successive extension of the Verma modules. For this purpose, we only need to show M_0 is a successive extension of 1-dimensional representations. This follows from the following exercise.

Exercise 2.1.9. Let M_0 be a finite-dimensional $\mathfrak{b}$-module such that the $\mathfrak{h}$-action is semisimple. Show that M_0 is a successive extension of 1-dimensional representations.

□

Corollary 2.1.10. *Let M be an object in $\mathcal{O}$.*

(1) *As an $\mathfrak{n}^-$-module, M is finitely generated.*
(2) *The $\mathfrak{n}$-action on M is locally nilpotent.*
(3) *Each weight subspace of M is finite-dimensional.*

2.2. Irreducible objects.

We can classify the irreducible objects in $\mathcal{O}$.

Proposition-Definition 2.2.1. *The Verma module M_λ admits a unique irreducible quotient module, which we denote by L_λ. The highest weight of L_λ is λ.*

Proof. Recall that M_λ is generated by a highest weight vector. It follows that any proper submodule $N \subseteq M_\lambda$ is a weight module such that λ is not a weight of N. It follows that the union N_{max} of all the proper submodules satisfies the same property. By construction, this is the unique maximal proper submodule of M_λ. Therefore $M_\lambda/N_{\mathsf{max}}$ is the unique irreducible quotient of M_λ.

□

Remark 2.2.2. When λ is integral and dominant, *a priori* we do not know L_λ defined above coincides with the finite-dimensional Weyl module in Theorem 1.2.1. This is because we have not yet proved that L_λ in Proposition-Definition 2.2.1 is finite-dimensional.

Exercise 2.2.3 (Problem 1, Homework 2)**.** For $\mathfrak{g} = \mathfrak{sl}_2$, let λ be a weight and l be the corresponding complex number. Show that the Verma module M_λ is irreducible iff $l \notin \mathbb{Z}^+$. For $l \in \mathbb{Z}^+$, show that there is a non-split short exact sequence

$$0 \to M_{\lambda'} \to M_\lambda \to L_\lambda \to 0 \tag{2.3}$$

such that L_λ is the finite-dimensional irreducible $\mathfrak{sl}_2$-module with highest weight λ. Moreover, the complex number l' that corresponds to λ' is equal to $-l-2$.

Theorem 2.2.4. *We have*

$$\mathsf{Irr}(\mathcal{O}) \simeq \{L_\lambda \,|\, \lambda \in \mathfrak{h}^*\}.$$

Proof. By definition, L_λ and $L_{\lambda'}$ are isomorphic iff $\lambda = \lambda'$. It remains to show that any irreducible object in $\mathcal{O}$ is isomorphic to L_λ for some weight λ.

Let $L \in \mathcal{O}$ be an irreducible object. By Proposition 2.1.8, there exists a surjective morphism $M \to L$ such that M can be written as a successive extension of Verma modules. We have a short exact sequence

$$0 \to M_\lambda \to M \to M' \to 0$$

in $\mathcal{O}$ such that M' can be written as a successive extension of strictly fewer Verma modules. Consider the composition

$$M_\lambda \to M \to L \tag{2.4}$$

If (2.4) is nonzero, it must be a surjection because L is irreducible. Then Proposition–Definition 2.2.1 implies $L \simeq L_\lambda$ as desired.

If (2.4) is zero, we obtain a surjection $M' \to L$. We can replace M with M' and finish the argument using induction.

□

Corollary 2.2.5. *When λ is integral and dominant, L_λ is finite-dimensional and isomorphic to the Weyl module with highest weight λ.*

Proof. This Weyl module is an irreducible object in $\mathcal{O}$ with highest weight λ. Theorem 2.2.4 implies it is isomorphic to L_λ.

□

2.3. **Harish–Chandra isomorphism.** We are going to study the center of the associative ring $U(\mathfrak{g})$. This is motivated by the following general paradigm in representation theory. Let U be an associative ring and $Z \subseteq U$ be its center. For an U-module M, we can view it as a Z-module and consider its *(set theoretic) support*

$$\mathsf{supp}_Z(M) := \{\mathfrak{p} \in \mathsf{Spec}(Z) \,|\, M_\mathfrak{p} \neq 0\}.$$

Let M and N are U-modules such that $\mathsf{supp}_Z(M) \bigcap \mathsf{supp}_Z(N) = \varnothing$. One can easily show that $\mathsf{Hom}_U(M,N) = 0$.

Notation 2.3.1. We write $Z(\mathfrak{g}) := Z(U(\mathfrak{g}))$ for the center of the associative ring $U(\mathfrak{g})$.

We are going to show that $Z(\mathfrak{g})$ is *non-canonically* isomorphic to a polynomial algebra in $\mathsf{rank}(\mathfrak{g}) := \dim(\mathfrak{h})$ variables. Let us first look at the $\mathfrak{sl}_2$-case.

Exercise 2.3.2 (Problem 2, Homework 2)**.** For $\mathfrak{g} = \mathfrak{sl}_2$, let $e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ and $h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the standard basis. Show that the *Casimir element*

$$\Omega := ef + fe + h^2/2 \in U(\mathfrak{sl}_2)$$

is contained in the center $Z(\mathfrak{sl}_2)$.

Exercise 2.3.3 (Problem 3, Homework 2)**.** For $\mathfrak{g} = \mathfrak{sl}_2$, show that the Casimir element Ω acts on the Verma module M_λ by the scaler $l + l^2/2$, where $l := \lambda(h)$.

Remark 2.3.4. We will see that $Z(\mathfrak{sl}_2)$ is isomorphic to $\mathbb{C}[\Omega]$. In particular, $Z(\mathfrak{sl}_2)$ acts on M_λ via a character that depends *algebraically* on λ. In other words, there is a *canonical* homomorphism

$$\phi : Z(\mathfrak{sl}_2) \to \mathcal{O}(\mathfrak{h}^*) \simeq \mathsf{Sym}(\mathfrak{h})$$

such that z acts on M_λ via the scaler $\phi(z)(\lambda)$. Namely, if we identify $\mathsf{Sym}(\mathfrak{h})$ with $\mathbb{C}[h]$, Exercise 2.3.3 implies $\phi(\Omega) = h + h^2/2$. Note that the homeomorphism ϕ is injective. Moreover, the image of ϕ is the subring preserved by the reflection $h \leftrightarrow -h-2$. Note that the last observation is compatible with Exercise 2.2.3.

With the above example in mind, let us start to study the general case.

Lemma 2.3.5. *The commutative ring $Z(\mathfrak{g})$ acts on the Verma module M_λ via a character, which we denote by $\xi_\lambda : Z(\mathfrak{g}) \to \mathbb{C}$.*

Proof. For $z \in Z(\mathfrak{g})$, the map $z \cdot - : M_\lambda \to M_\lambda$ is $U(\mathfrak{g})$-linear. In particular, it preserves the weight subspaces of M_λ. Let $v_\lambda \in M_\lambda$ be a nonzero highest weight vector. It follows that $\mathbb{C} \cdot v_\lambda \subseteq M_\lambda$ is preserved by the action of z, hence there exists a unique scaler $\xi_\lambda(z) \in \mathbb{C}$ such that

$$z \cdot v_\lambda = \xi_\lambda(z) v_\lambda.$$

Moreover, $\xi_\lambda(-) : Z(\mathfrak{g}) \to \mathbb{C}$ is a character by the axioms of an action.

It remains to show that $Z(\mathfrak{g})$ acts on any vector v of M_λ via the character ξ_λ constructed as above. Recall that v can be written as $u \cdot v_\lambda$ for some element $u \in U(\mathfrak{g})$. Then we have

$$z \cdot v = z \cdot (u \cdot v_\lambda) = u \cdot (z \cdot v_\lambda) = u \cdot (\xi_\lambda(z) v_\lambda) = \xi_\lambda(z)(u \cdot v_\lambda) = \xi_\lambda(z) v$$

as desired.

□

In the proof of the above lemma, we constructed ξ_λ by studying the action of $Z(\mathfrak{g})$ on a highest weight vector $v_\lambda \in \mathbb{C}_\lambda \subseteq M_\lambda$. In other words, it is equal to the composition

$$Z(\mathfrak{g}) \to U(\mathfrak{g}) \xrightarrow{-\cdot v_\lambda} M_\lambda \to \mathbb{C}_\lambda \simeq \mathbb{C},$$

where the third map is the projection to the highest weight vector, while the last isomorphism sends $v_\lambda \in \mathbb{C}_\lambda$ to $1 \in \mathbb{C}$. Using the identification $M_\lambda \simeq U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda$, we can rewrite the above composition as

$$Z(\mathfrak{g}) \to U(\mathfrak{g}) \to U(\mathfrak{g}) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \simeq (U(\mathfrak{n}^-) \otimes U(\mathfrak{b})) \underset{U(\mathfrak{b})}{\otimes} \mathbb{C}_\lambda \to \mathbb{C}_\lambda \simeq \mathbb{C},$$

where the fourth map is given by the augmentation map $U(\mathfrak{n}^-) \to \mathbb{C}$ that sends any element $f \in \mathfrak{n}^-$ to zero. We can further rewrite the above composition as

$$Z(\mathfrak{g}) \to U(\mathfrak{g}) \to \mathbb{C} \underset{U(\mathfrak{n}^-)}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C} \simeq U(\mathfrak{h}) \simeq \mathsf{Sym}(\mathfrak{h}) \xrightarrow{-\cdot\lambda} \mathbb{C},$$

where the second map is given by the augmentation maps of $U(\mathfrak{n}^-)$ and $U(\mathfrak{n})$.

This motivates the following result.

Proposition 2.3.6. *The composition*

$$\phi : Z(\mathfrak{g}) \to U(\mathfrak{g}) \to \mathbb{C} \underset{U(\mathfrak{n}^-)}{\otimes} U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C} \simeq U(\mathfrak{h}) \simeq \mathsf{Sym}(\mathfrak{h})$$

is a ring homomorphism.

Proof. By the previous discussion, for any $\lambda \in \mathfrak{h}^*$, the composition

$$Z(\mathfrak{g}) \xrightarrow{\phi} \mathsf{Sym}(\mathfrak{h}) \xrightarrow{-\cdot\lambda} \mathbb{C}$$

is equal to the character ξ_λ. In particular, it is a ring homomorphism. This implies that ϕ is a ring homomorphism because the homomorphism

$$\mathsf{Sym}(\mathfrak{h}) \to \prod_{\lambda\in\mathfrak{h}^*} \mathbb{C},\ r \mapsto r(\lambda)$$

is injective.

□

Warning 2.3.7. One can view $\mathsf{Sym}(\mathfrak{h})$ as a *subring* of $U(\mathfrak{g})$, but $Z(\mathfrak{g})$ is not contained in this subring. This can be seen from the $\mathfrak{sl}_2$-case.

Remark 2.3.8. By definition, $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})$ is compatible with the natural filtrations on both sides.

Let

$$\varpi : \mathfrak{h}^* \simeq \mathsf{Spec}(\mathsf{Sym}(\mathfrak{h})) \to \mathsf{Spec}(Z(\mathfrak{g}))$$

be the morphism between affine schemes that corresponds to the Harish-Chandra homomorphism ϕ. The previous discussion can be reformulated as the following result.

Corollary 2.3.9. *The scheme theoretic support of M_λ, viewed as a $Z(\mathfrak{g})$-module, is equal to $\{\varpi(\lambda)\}$.*

As exhibited by the $\mathfrak{sl}_2$-case, the homomorphism $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})$ is not an isomorphism. Instead, we expect it to be an injection with image equal to a certain invariant subring of $\mathsf{Sym}(\mathfrak{h})$.

Definition 2.3.10. The *dot action* of the Weyl group W on $\mathfrak{h}^*$ is defined to be

$$w \cdot \lambda \coloneqq w(\lambda + \rho) - \rho,\ \ w \in W,\ \lambda \in \mathfrak{h}^*$$

The *dot action* of W on $\mathsf{Sym}(\mathfrak{h})$ is defined to be the action induced from the isomorphism $\mathsf{Sym}(\mathfrak{h}) \simeq \mathcal{O}(\mathfrak{h}^*)$.

Remark 2.3.11. Note that $-\rho$ is the unique fixed point for the dot W-action on $\mathfrak{h}^*$.

Remark 2.3.12. By definition, for $w \in W$, $\lambda \in \mathfrak{h}^*$ and $r \in \mathsf{Sym}(\mathfrak{h})$, we have

$$(w \cdot r)(\lambda) = r(w^{-1} \cdot \lambda) = r(w^{-1}(\lambda + \rho) - \rho).$$

Example 2.3.13. For $\mathfrak{g} = \mathfrak{sl}_2$, the dot action of the unique non-trivial element $s \in W$ on $\mathfrak{h}^*$ is given by $l \mapsto -l - 2$. The dot action of s on $\mathsf{Sym}(\mathfrak{h})$ is given by $h \mapsto -h - 2$.

Theorem 2.3.14 (Harish-Chandra)**.** *The homomorphism $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})$ induces an isomorphism*

$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \xrightarrow{\simeq} \mathsf{Sym}(\mathfrak{h})^{W_\bullet}.$$

The rest of this subsection is devoted to the proof of the theorem.

Lemma 2.3.15. *Let $\alpha \in \Delta$ be a simple root and $\lambda \in \mathfrak{h}^*$. Suppose that $\langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^+$. Then M_λ contains $M_{s_\alpha \cdot \lambda}$ as a submodule.*

Proof. Recall that

$$s_\alpha(\lambda) = \lambda - 2\frac{(\lambda,\alpha)}{(\alpha,\alpha)}\alpha = \lambda - \langle \lambda, \check{\alpha}\rangle \alpha.$$

It follows that

$$s_\alpha \cdot \lambda = \lambda - \langle \lambda + \rho, \check{\alpha}\rangle \alpha = \lambda - m\alpha$$

for $m \in \mathbb{Z}^{\geq 0}$. The lemma is obvious for $m = 0$. We assume $m > 0$. Then we have $\langle \lambda, \check{\alpha}\rangle = m-1$ because $\langle \rho, \check{\alpha}\rangle = 1$.

Let $f_\alpha \in \mathfrak{g}_{-\alpha}$ be a generator. Note that $f_\alpha^m \cdot v_\lambda$ is a weight vector whose weight is equal to $\lambda - m\alpha = s_\alpha \cdot \lambda$. We claim that

$$\mathfrak{n} \cdot (f_\alpha^m \cdot v_\lambda) = 0. \tag{2.5}$$

Assuming the claim, the map $\mathbb{C}_{s_\alpha \cdot \lambda} \to M_\lambda,\ c \mapsto c(f_\alpha^m \cdot v_\lambda)$ is $\mathfrak{b}$-linear, and thereby induces a $\mathfrak{g}$-linear map $M_{s_\alpha \cdot \lambda} \to M_\lambda$. This map is injective because as a morphism between $U(\mathfrak{n}^-)$-modules, it is given by $- \cdot f_\alpha^m : U(\mathfrak{n}^-) \to U(\mathfrak{n}^-)$, which is injective by the PBW theorem.

It remains to prove (2.5). For each simple root $\beta \in \Delta$, let $e_\beta \in \mathfrak{n}$ be a nonzero vector of weight β. Note that the vectors $(e_\beta)_{\beta\in\Delta}$ generate $\mathfrak{n}$ under Lie brackets. Hence we only need to show $e_\beta \cdot f_\alpha^m \cdot v_\lambda = 0$.

If $\alpha \neq \beta$, we have $[e_\beta, f_\alpha] = 0$ because $\beta - \alpha$ is neither a root nor zero. It follows that

$$e_\beta \cdot f_\alpha^m \cdot v_\lambda = f_\alpha^m \cdot e_\beta \cdot v_\lambda = 0$$

as desired.

If $\alpha = \beta$, we can resacle e_α such that $[e_\alpha, f_\alpha] = \check{\alpha}$. Then

$$[\check{\alpha}, f_\alpha] = \langle -\alpha, \check{\alpha}\rangle f_\alpha = -2f_\alpha.$$

We have

$$\begin{aligned} e_\alpha \cdot f_\alpha^m \cdot v_\lambda &= \sum_{1\leq i\leq m} f_\alpha^{m-i} \cdot [e_\alpha, f_\alpha] \cdot f_\alpha^{i-1} \cdot v_\lambda + f_\alpha^m \cdot e_\alpha \cdot v_\lambda \\ &= \sum_{1\leq i\leq m} f_\alpha^{m-i} \cdot \check{\alpha} \cdot f_\alpha^{i-1} \cdot v_\lambda \end{aligned}$$

Note that

$$\check{\alpha} \cdot f_\alpha^j = \sum_{1\leq i\leq j} f_\alpha^{j-i} \cdot [\check{\alpha}, f_\alpha] \cdot f_\alpha^{i-1} + f_\alpha^j \cdot \check{\alpha} = -2j f_\alpha^j + f_\alpha^j \cdot \check{\alpha}.$$

Hence

$$e_\alpha \cdot f_\alpha^m \cdot v_\lambda = \sum_{1\leq i\leq m} (-2(i-1)f_\alpha^{m-1} + f_\alpha^{m-1} \cdot \check{\alpha})v_\lambda = \big(-m(m-1) + m\langle \lambda, \check{\alpha}\rangle\big)v_\lambda = 0$$

as desired.

□

Lemma 2.3.16. *The image of* $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})$ *is contained in* $\mathsf{Sym}(\mathfrak{h})^{W\bullet}$.

Proof. We only need to show that for any $w \in W$, $z \in Z(\mathfrak{g})$ and $\lambda \in \mathfrak{h}^*$,

$$\phi(z)(\lambda) = \phi(z)(w \cdot \lambda). \tag{2.6}$$

Since W is generated by by simple reflections, we only need to check (2.6) for $w = s_\alpha,\ \alpha \in \Delta$.

Let $\alpha \in \Delta$ be a fixed simple root. It is easy to see that

$$\{\lambda \in \mathfrak{h}^* \mid \langle \lambda + \rho, \check{\alpha} \rangle \in \mathbb{Z}^+\}$$

is a Zariski dense subset of $\mathfrak{h}^*$. Hence we only need to check (2.6) for λ belonging to this set. By Lemma 2.3.15, we have $M_{s_\alpha \cdot \lambda} \subseteq M_\lambda$. This implies (2.6) because z acts on M_μ via the scaler $\phi(z)(\mu)$.

□

The above lemma implies we have a well-defined homomorphism

$$\phi_{\mathsf{HC}} : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})^{W_\bullet}$$

between filtered commutative rings, where the filtrations on the LHS and the RHS are induced by the PBW filtrations on $U(\mathfrak{g})$ and $\mathsf{Sym}(\mathfrak{h})$ respectively. As a consequence, it induces a homomorphism

$$\mathsf{gr}^\bullet \phi_{\mathsf{HC}} : \mathsf{gr}^\bullet Z(\mathfrak{g}) \to \mathsf{gr}^\bullet(\mathsf{Sym}(\mathfrak{h})^{W_\bullet}). \tag{2.7}$$

To show that ϕ_{HC} is an isomorphism, we only need to show $\mathsf{gr}^\bullet \phi_{\mathsf{HC}}$ is so.

Lemma 2.3.17. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathsf{gr}^\bullet Z(\mathfrak{g}) & \overset{\simeq}{\dashrightarrow} & \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \\ \downarrow{\scriptstyle \subseteq} & & \downarrow{\scriptstyle \subseteq} \\ \mathsf{gr}^\bullet U(\mathfrak{g}) & \underset{\mathsf{PBW}}{\overset{\simeq}{\longrightarrow}} & \mathsf{Sym}(\mathfrak{g}), \end{array}$$

where $\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}}$ *is the invariance of* $\mathsf{Sym}(\mathfrak{g})$ *with respect to the* adjoint action[1].

Proof. Consider the adjoint action of $\mathfrak{g}$ on $U(\mathfrak{g})$ given by

$$\mathfrak{g} \times U(\mathfrak{g}) \to U(\mathfrak{g}), \ \ (x, u) \mapsto [x, u] := xu - ux.$$

One can check that this action preserves the PBW filtration on $U(\mathfrak{g})$, and the induced action on $\mathsf{gr}^\bullet(U(\mathfrak{g})) \simeq \mathsf{Sym}(\mathfrak{g})$ is given by the adjoint $\mathfrak{g}$-action on $\mathsf{Sym}(\mathfrak{g})$.

Note that for any $n \in \mathbb{Z}^+$, the $\mathfrak{g}$-module $\mathsf{F}^{\leq n} U(\mathfrak{g})$ is finite-dimensional. This implies the short exact sequence

$$0 \to \mathsf{F}^{\leq n-1} U(\mathfrak{g}) \to \mathsf{F}^{\leq n} U(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g}) \to 0$$

splits. Taking $\mathfrak{g}$-invariance, we obtain a split short exact sequence

$$0 \to \mathsf{F}^{\leq n-1} Z(\mathfrak{g}) \to \mathsf{F}^{\leq n} Z(\mathfrak{g}) \to \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}} \to 0.$$

This gives the desired isomorphism $\mathsf{gr}^n Z(\mathfrak{g}) \simeq \mathsf{Sym}^n(\mathfrak{g})^{\mathfrak{g}}$.

□

A similar proof gives the following result.

Lemma 2.3.18. *There is a unique dotted graded isomorphism making the following diagram commute*

$$\begin{array}{ccc} \mathsf{gr}^\bullet(\mathsf{Sym}(\mathfrak{h})^{W_\bullet}) & \overset{\simeq}{\dashrightarrow} & \mathsf{Sym}(\mathfrak{h})^{W} \\ \downarrow{\scriptstyle \subset} & & \downarrow{\scriptstyle \subset} \\ \mathsf{gr}^\bullet \mathsf{Sym}(\mathfrak{h}) & \overset{\simeq}{\longrightarrow} & \mathsf{Sym}(\mathfrak{h}), \end{array}$$

[1]This is the unique action that restricts to the adjoint action on $\mathfrak{g} \subseteq \mathsf{Sym}(\mathfrak{g})$ and satisfies the Lebniz rule.

where the right-top corner is the invariance for the (usual) linear *W-action on* $\mathsf{Sym}(\mathfrak{h})$.

By Lemma 2.3.17 and Lemma 2.3.18, the homomorphism (2.7) can be identified with a certain homomorphism

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{h})^{W},$$

and we only need to show ϕ_{cl} is an isomorphism.

We are going to give a direct description for the homomorphism ϕ_{cl}. For this purpose, we give the following alternative construction of ϕ.

Exercise 2.3.19 (Problem 4, Homework 2)**.** Consider the adjoint $\mathfrak{h}$-action on $U(\mathfrak{g})$. Show that it preserves the kernel of the surjection

$$U(\mathfrak{g}) \to U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C}$$

and induces an $\mathfrak{h}$-action on $U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}$. Moreover, we have

$$(U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C})^{\mathfrak{h}} \simeq U(\mathfrak{b}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C} \simeq \mathsf{Sym}(\mathfrak{h}).$$

Exercise 2.3.20 (Problem 5, Homework 2)**.** Show that $\phi : Z(\mathfrak{g}) \to \mathsf{Sym}(\mathfrak{h})$ can be identified with the composition

$$Z(\mathfrak{g}) \overset{\subseteq}{\to} U(\mathfrak{g})^{\mathfrak{h}} \to (U(\mathfrak{g}) \underset{U(\mathfrak{n})}{\otimes} \mathbb{C})^{\mathfrak{h}} \simeq \mathsf{Sym}(\mathfrak{h}),$$

where the last isomorphism is due to Exercise 2.3.19.

It follows that ϕ_{HC} is the unique map that fits into the following commutative diagram

$$\begin{array}{ccc} Z(\mathfrak{g}) & \overset{\phi_{\mathsf{HC}}}{\dashrightarrow} & \mathsf{Sym}(\mathfrak{h})^{W\bullet} \\ \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} \\ U(\mathfrak{g}) & \longrightarrow & U(\mathfrak{g}) \otimes_{U(\mathfrak{n})} \mathbb{C}. \end{array}$$

Taking associated graded pieces, we deduce that ϕ_{cl} is the unique map that fits into the following diagram

$$\begin{array}{ccc} \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} & \overset{\phi_{\mathsf{cl}}}{\dashrightarrow} & \mathsf{Sym}(\mathfrak{h})^{W} \\ \downarrow{\scriptstyle\subseteq} & & \downarrow{\scriptstyle\subseteq} \\ \mathsf{Sym}(\mathfrak{g}) & \longrightarrow & \mathsf{Sym}(\mathfrak{g}/\mathfrak{n}). \end{array}$$

Remark 2.3.21. Let G be a connected algebraic group with Lie algebra $\mathfrak{g}$. Note that we have $\mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} = \mathsf{Sym}(\mathfrak{g})^{G}$. The previous discussion implies there is a unique dotted morphism between schemes that fits into the following diagram

$$\begin{array}{ccc} \mathfrak{g}^*/\!/G & \dashleftarrow & \mathfrak{h}^*/\!/W \\ \uparrow & & \uparrow \\ \mathfrak{g}^* & \longleftarrow & (\mathfrak{g}/\mathfrak{n})^*. \end{array}$$

Here $\mathfrak{g}^*/\!/G$ and $\mathfrak{h}^*/\!/W$ are the (GIT) quotient schemes.

Recall that the Killing form induces a G-linear isomorphism $\mathfrak{g} \simeq \mathfrak{g}^*$ and a W-linear isomorphism $\mathfrak{h} \simeq \mathfrak{h}^*$. Via the first isomorphism, $\mathfrak{b} \subseteq \mathfrak{g}$ corresponds to $(\mathfrak{g}/\mathfrak{n})^*$. It follows that there is a unique dotted morphsim between schemes that fits into the following diagram

$$\begin{array}{ccc} \mathfrak{g}/\!/G & \leftarrow\cdots & \mathfrak{h}/\!/W \\ \uparrow & & \uparrow \\ \mathfrak{g} & \longleftarrow & \mathfrak{b}, \end{array}$$

where the right vertical morphism is the composition $\mathfrak{b} \to \mathfrak{h} \to \mathfrak{h}/\!/W$.

Note that the similar claim where $\mathfrak{b}$ is replaced with $\mathfrak{h}$ is true but weaker than the above claim.

As a byproduct of the above observation, one can prove the following result.

Exercise 2.3.22 (Problem 6, Homework 2)**.** Let $r \in \mathcal{O}(\mathfrak{g})^G$ be a G-invariant regular function on $\mathfrak{g}$. Show that $r(x) = r(0)$ for any nilpotent element $x \in \mathfrak{g}$.

Recall that we only need to show that

$$\phi_{\mathsf{cl}} : \mathsf{Sym}(\mathfrak{g})^{\mathfrak{g}} \to \mathsf{Sym}(\mathfrak{h})^W$$

is an isomorphism. By Remark 2.3.21, this is equivalent to the following result.

Theorem 2.3.23 (Chevalley)**.** *The restriction map $\mathcal{O}(\mathfrak{g}) \to \mathcal{O}(\mathfrak{h})$ induces an isomorphism*

$$\phi_{\mathsf{Chev}} : \mathcal{O}(\mathfrak{g})^{\mathfrak{g}} \xrightarrow{\simeq} \mathcal{O}(\mathfrak{h})^W.$$

Proof. By the previous discussions, the restriction map induces a homomorphism $\phi_{\mathsf{Chev}} : \mathcal{O}(\mathfrak{g})^{\mathfrak{g}} \to \mathcal{O}(\mathfrak{h})^W$. This homomorphism is injective because semisimple elements form a Zariski dense subset of $\mathfrak{g}$, and any such element is conjugate to an element in $\mathfrak{h}$.

It remains to show ϕ_{Chev} is surjective. Recall that the subset Λ^+ of dominant integral weights spans $\mathfrak{h}^*$ as a vector space. This implies that for any $n \geq 0$, the subset $\{\lambda^n \,|\, \lambda \in \Lambda^+\}$ spans $\mathsf{Sym}^n(\mathfrak{h}^*)$. Therefore the elements

$$b_{\lambda,n} := \sum_{w \in W} w(\lambda^n), \quad \lambda \in \Lambda^+, n \geq 0$$

span $\mathsf{Sym}(\mathfrak{h}^*)^W \simeq \mathcal{O}(\mathfrak{h})^W$.

It remains to show that each element $b_{\lambda,n}$ defined as above is contained in the image of ϕ_{Chev}. Consider the elements $a_{\lambda,n} \in \mathcal{O}(\mathfrak{g})$ defined by the formula

$$a_{\lambda,n}(x) := \mathsf{tr}(x^n, L_\lambda), \quad \forall x \in \mathfrak{g}.$$

Using the fact that L_λ is G-integrable for simply connected G, we see that $a_{\lambda,n} \in \mathcal{O}(\mathfrak{g})^G = \mathcal{O}(\mathfrak{g})^{\mathfrak{g}}$.

Exercise 2.3.24 (Problem 7, Homework 2)**.** Prove that

$$\phi_{\mathsf{Chev}}(a_{\lambda,n}) = \frac{1}{\#\mathsf{Stab}_W(\lambda)} b_{\lambda,n} \mod \mathsf{span}(b_{\lambda',n},\ \lambda' < \lambda),$$

where $\mathsf{Stab}_W(\lambda) \subseteq W$ is the stablizer of the W-action at $\lambda \in \mathfrak{h}^*$.

Now one can finish the proof by using the above exercise and induction in λ.

□

□[Theorem 2.3.14]

References

[H1] Humphreys, James E. Introduction to Lie algebras and representation theory. Vol. 9. Springer Science & Business Media, 2012.

[H2] Humphreys, James E. Representations of Semisimple Lie Algebras in the BGG Category $\mathcal{O}$. Vol. 94. American Mathematical Soc., 2008.

[MR] McConnell, John C., James Christopher Robson, and Lance W. Small. Noncommutative noetherian rings. Vol. 30. American Mathematical Soc., 2001.